

1. Let $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 2 & -2 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$.

(a) Find a row echelon form of A .

(b) Find the reduced row echelon form of A .

2. Each of the following matrices is the augmented matrix of a linear system. Write the solutions in parametric vector form for each system which has a solution. Write no solution if applicable.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{array} \right]$ (b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right]$ (c) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

3. Solve the following system of linear equations. Write down which elementary row operations you used at each stage. If there are an infinite number of solutions, give the complete solutions in terms of parameter(s) and also state one particular solution.

$$\begin{aligned} x_1 - 2x_2 - 3x_4 &= 2 \\ -2x_1 + 3x_2 + 3x_3 + 2x_4 &= -2 \\ x_2 - x_3 + 4x_4 &= 2. \end{aligned}$$

$$\begin{aligned} x - 4y + 5z &= -3 \\ 3x + y + 2z &= 4 \\ 2x + 5y - 3z &= 7 \end{aligned}$$

$$\begin{aligned} x + y + z &= 2 \\ y - 2z + w &= 0 \\ 2y + z + 4w &= 3 \\ x - y + 4z - 2w &= 1 \end{aligned}$$

4. Consider the following system of linear equations.

$$\begin{aligned}x + y + 4z &= a \\x - z &= b \\x - 2y - 11z &= c.\end{aligned}$$

Describe the set of all $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for which the system is consistent.

5. Consider the following augmented matrices.

$$\begin{aligned}\text{(i)} \quad & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & (a-4)(a+3) & a-4 \end{array} \right], & \text{(ii)} \quad & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & a+3 & a-4 \end{array} \right], \\ \text{(iii)} \quad & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & a+3 & (a+3)(a-4) \end{array} \right], & \text{(iv)} \quad & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & (a+3)(a-4) \end{array} \right].\end{aligned}$$

For each augmented matrix, determine the value(s) of a such that the corresponding system has

- (a) exactly one solution,
- (b) infinitely many solutions,
- (c) no solutions.

6. Consider the following systems of linear equations.

$$\begin{aligned}\text{(i)} \quad & \begin{aligned}x - 5y &= 3 \\ 3x - 8y &= -5 \\ -x + 2y &= k\end{aligned} & \text{(ii)} \quad & \begin{aligned}x - 6y + 2z &= 8 \\ 4x - 25y + 10z &= 6 \\ 2x - 13y + 6z &= k\end{aligned}\end{aligned}$$

For each system, determine the value(s) of k so that the system will have

- (a) no solutions, (b) a unique solution, (c) infinitely many solutions.

7. Write the following system in matrix form $Ax = b$ and express the general solution as a sum of a particular solution and the general solution of the associated homogeneous system $Ax = 0$.

$$\begin{array}{rrrrrrcl} 2x_1 & + & x_2 & - & x_3 & - & x_4 & = & -1 \\ 3x_1 & + & x_2 & + & x_3 & - & 2x_4 & = & -2 \\ -x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 2 \\ -2x_1 & - & x_2 & & & + & 2x_4 & = & 3 \end{array}$$

8. Let $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Give an example of a 3×3 matrix A for each of the following cases:

- (a) $AX = b$ has no solutions,
- (b) $AX = b$ has unique solution,
- (c) $AX = b$ has infinitely many solutions.

9. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Find, if possible, a 3×1 vector b such that

$AX = b$ has no solutions. If not possible, explain your answer.

10. (a) Solve the vector equation: $x \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

- (b) Solve the matrix equation: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.